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ON THE PROBLEM OF STEADY MOTIONS STABILITY OF NONHOLONOMIC SYSTEMS*

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The results of investigations of stability of steady motions of Chaplygin's nonholonomic system /1/ are extended to the case of nonholonomic systems of a general form. The possibility of asymptotic stability with respect to a part of variables of steady motions of conservative nonholonomic systems is indicated. The results are illustrated by an example.

1. We shall consider a scleronomic nonholonomic mechanical system subjected to the action of potential and dissipative forces. We assume the velocities $q_1^\cdot, \ldots, q_n^\cdot$ of generalized coordinates $q_1, \ldots, q_n$ to be related by $n-l$ nonintegrable relations of the form

$$q_\varkappa^\cdot = \sum_{r=1}^{l} b_{\varkappa r}(q)\, q_r^\cdot \quad (\varkappa = l+1, \ldots, n) \tag{1.1}$$

and use the equations of motion of the system in the Voronets form

$$\frac{d}{dt}\frac{\partial \Theta}{\partial q_r^\cdot} = \frac{\partial(\Theta+U)}{\partial q_r} + \sum_{\varkappa=l+1}^{n} \frac{\partial(\Theta+U)}{\partial q_\varkappa} b_{\varkappa r} + \sum_{\varkappa=l+1}^{n} \Theta_\varkappa \sum_{s=1}^{l} \nu_{\varkappa rs} q_s^\cdot - \frac{\partial \Phi}{\partial q_r^\cdot} \quad (r = 1, \ldots, l) \tag{1.2}$$

$$\nu_{\varkappa rs} = \frac{\partial b_{\varkappa r}}{\partial q_s} - \frac{\partial b_{\varkappa s}}{\partial q_r} + \sum_{\lambda=l+1}^{n} \left(b_{\lambda s}\frac{\partial b_{\varkappa r}}{\partial q_\lambda} - b_{\lambda r}\frac{\partial b_{\varkappa s}}{\partial q_\lambda} \right)$$

where

$$2\Theta = \sum_{r,s=1}^{l} a_{rs}(q)\, q_r^\cdot q_s^\cdot, \quad \Theta_\varkappa = \sum_{p=1}^{l} \theta_{\varkappa p}(q)\, q_p^\cdot, \qquad 2\Phi = \sum_{r,s=1}^{l} f_{rs}(q)\, q_r^\cdot q_s^\cdot$$

which were obtained by eliminating the quantities $q_\varkappa^\cdot$ from $2T, \partial T / \partial q_\varkappa^\cdot$, and $2F$, respectively, using formulas (1.1); T is the kinetic energy of the system; F is a dissipative function, and U a force function.

We assume that conditions

$$\frac{\partial(T+U)}{\partial q_\mu} = 0, \quad \frac{\partial F}{\partial q_\mu} = 0, \quad \frac{\partial b_{\varkappa r}}{\partial q_\mu} = 0 \tag{1.3}$$

$$\frac{\partial(\Theta+U)}{\partial q_\alpha} = 0, \quad \frac{\partial \Phi}{\partial q_\alpha} = 0, \quad \frac{\partial b_{\rho r}}{\partial q_\alpha} = 0, \qquad \frac{\partial}{\partial q_\alpha} \sum_{\varkappa=l+1}^{n} \theta_{\varkappa p} \nu_{\varkappa rs} = 0 \tag{1.4}$$

$$(\mu = m+1, \ldots, n \; (m \geqslant l); \; \alpha = k+1, \ldots, l; \; \rho = l+1, \ldots, m$$
$$p, r, s = 1, \ldots, l; \; \varkappa = l+1, \ldots, n)$$

are satisfied.

Conditions (1.3) imply that Eqs. (1.2) can be considered independently of the last $n-m$ equations of nonholonomic relations which represent Chaplygin type relationships (the first $m-l$ nonholonomic relations are relations of the general form; the case of $m = l$ was considered in /1/). Conditions (1.4) imply that q_α $(\alpha = k+1, \ldots, l)$ are ignorable coordinates in conformity with the definition in /2/.

We moreover assume that

$$\frac{\partial \Phi}{\partial q_\alpha^\cdot} = 0, \quad \sum_{\mu=m+1}^{n} \theta_{\mu\beta} \nu_{\mu\alpha\gamma} = 0, \quad b_{\rho\alpha} = 0, \qquad (\alpha, \beta, \gamma = k+1, \ldots, l; \;\; \rho = l+1, \ldots, m) \tag{1.5}$$

The first group of conditions (1.5) indicates the absence of dissipation by cyclic velocities, and the second and third groups ensure the existence of manifold of steady motions of a dimension not less than the sum of ignorable coordinates and the number of nonholonomic relations of the general form. (The third groups of conditions (1.4) and (1.5) must be omitted in Chaplygin's systems, while the second group of conditions (1.5) is necessarily satisfied in the case of a single ignorable coordinate).

On the above assumptions the equations of motion (1.2) together with the equations of

*Prikl.Matem.Mekhan.,44,No.3,418-426,1980 nonholonomic relations of the general form obviously admit the solution

$$q_i = q_{i0}, \quad q_i^{\cdot} = 0 \ (i = 1, \ldots, k), \quad q_\alpha^{\cdot} = q_{\alpha 0}^{\cdot} \ (\alpha = k+1, \ldots, l); \ q_\rho = q_{\rho 0} \quad (\rho = l+1, \ldots, m) \quad (1.6)$$

and the m constants $q_{i0}, q_{\alpha 0}^{\cdot}, q_{\rho 0}$ satisfy the system of $k < m$ equations

$$\frac{\partial U}{\partial q_i} + \sum_{\rho=l+1}^{m} \frac{\partial U}{\partial q_\rho} b_{\rho i} + \sum_{\alpha,\beta=k+1}^{l} \left[\frac{1}{2}\left(\frac{\partial a_{\alpha\beta}}{\partial q_i} + \sum_{\rho=l+1}^{m} \frac{\partial a_{\alpha\beta}}{\partial q_\rho} b_{\rho i}\right) + \sum_{\mu=m+1}^{n} \theta_{\mu\alpha}\nu_{\mu i\beta}\right] q_\alpha^{\cdot} q_\beta^{\cdot} = 0 \quad (i = 1, \ldots, k) \quad (1.7)$$

Note that nonholonomic systems with ignorable coordinates in the sense of definition (1.4) and (1.5) do not generally admit cyclic integrals.

2. Let us consider an arbitrary point of the manifold (1.7) and pose the question of stability of solution (1.6) with respect to perturbations of variables $q_i, q_i^{\cdot}, q_\alpha^{\cdot}$, and q_ρ. Setting

$$x_i = q_i - q_{i0}, \quad y_\alpha = q_\alpha^{\cdot} - \omega_\alpha \ (\omega_\alpha = q_{\alpha 0}^{\cdot}), \quad z_\rho = q_\rho - q_{\rho 0}$$

we obtain the equations of perturbed motion

$$\sum_j a_{ij} x_j^{\cdot\cdot} + \sum_\beta a_{i\beta} y_\beta^{\cdot} = \sum_{j,h} B_{ijh} x_j^{\cdot} x_h^{\cdot} + \sum_{j,\beta} B_{ij\beta} x_j^{\cdot} (\omega_\beta + y_\beta) + \quad (2.1)$$
$$\sum_{\beta,\gamma} \omega_\beta\omega_\gamma \Delta B_{i\beta\gamma} + \sum_{\beta,\gamma} (\omega_\beta y_\gamma + \omega_\gamma y_\beta + y_\beta y_\gamma) B_{i\beta\gamma} + \Delta\left(\frac{\partial U}{\partial q_i} + \sum_\rho \frac{\partial U}{\partial q_\rho} b_{\rho i}\right) - \sum_j f_{ij} x_j^{\cdot}$$
$$\sum_j a_{\alpha j} x_j^{\cdot\cdot} + \sum_\beta a_{\alpha\beta} y_\beta^{\cdot} = \sum_{j,h} B_{\alpha jh} x_j^{\cdot} x_h^{\cdot} + \sum_{j,\beta} B_{\alpha j\beta} x_j^{\cdot} (\omega_\beta + y_\beta)$$
$$z_\rho^{\cdot} = \sum_j b_{\rho j} x_j^{\cdot}$$
$$B_{ijh} = \frac{1}{2}\frac{\partial a_{jh}}{\partial q_i} - \frac{\partial a_{ij}}{\partial q_h} + \sum_\rho \left(\frac{1}{2}\frac{\partial a_{jh}}{\partial q_\rho} b_{\rho i} - \frac{\partial a_{ij}}{\partial q_\rho} b_{\rho h}\right) + \sum_\varkappa \theta_{\varkappa h}\nu_{\varkappa ij}$$
$$B_{ij\beta} = \frac{\partial a_{j\beta}}{\partial q_i} - \frac{\partial a_{i\beta}}{\partial q_j} + \sum_\rho \left(\frac{\partial a_{j\beta}}{\partial q_\rho} b_{\rho i} - \frac{\partial a_{i\beta}}{\partial q_\rho} b_{\rho j}\right) + \sum_\varkappa (\theta_{\varkappa\beta}\nu_{\varkappa ij} + \theta_{\varkappa j}\nu_{\varkappa i\beta})$$
$$B_{i\beta\gamma} = \frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial q_i} + \sum_\rho \frac{\partial a_{\beta\gamma}}{\partial q_\rho} b_{\rho i}\right) + \sum_\mu \theta_{\mu\beta}\nu_{\mu i\gamma}$$
$$B_{\alpha jh} = \sum_\mu \theta_{\mu h}\nu_{\mu\alpha j} - \left(\frac{\partial a_{\alpha j}}{\partial q_h} + \sum_\rho \frac{\partial a_{\alpha j}}{\partial q_\rho} b_{\rho h}\right)$$
$$B_{\alpha j\beta} = \sum_\mu (\theta_{\mu\beta}\nu_{\mu\alpha j} + \theta_{\mu j}\nu_{\mu\alpha\beta}) - \left(\frac{\partial a_{\alpha\beta}}{\partial q_i} + \sum_\rho \frac{\partial a_{\alpha\beta}}{\partial q_\rho} b_{\rho j}\right)$$

Here and subsequently $i, j, h = 1, \ldots, k$; $\alpha, \beta, \gamma = k+1, \ldots, l$; $\rho, \sigma = l+1, \ldots, m$; $\mu = m+1, \ldots, n$, and $\varkappa = l+1, \ldots, n$. All coeffficients of system (2.1) are calculated for $q_i = q_{i0} + x_i$ and $q_\rho = q_{\rho 0} + z_\rho$; and the symbol Δ denotes the remainder of values of the respective functions at points $(q_{i0} + x_i, q_{\rho 0} + z_\rho)$ and $(q_{i0}, q_{\rho 0})$.

In the neighborhood of solution (1.6) the first approximation equations assume the form

$$\sum_j a_{ij}^{\circ} x_j^{\cdot\cdot} + \sum_\beta a_{i\beta}^{\circ} y_\beta^{\cdot} = \sum_j u_{ij}^{\circ} x_j + \sum_j v_{ij}^{\circ} x_j^{\cdot} + \sum_\beta u_{i\beta}^{\circ} y_\beta + \sum_\sigma u_{i\sigma}^{\circ} z_\sigma \quad (2.2)$$
$$\sum_j a_{\alpha j}^{\circ} x_j^{\cdot\cdot} + \sum_\beta a_{\alpha\beta}^{\circ} y_\beta^{\cdot} = \sum_j v_{\alpha j}^{\circ} x_j^{\cdot}$$
$$z_\rho^{\cdot} = \sum_j b_{\rho j}^{\circ} x_j^{\cdot}$$
$$u_{ij} = \frac{\partial}{\partial q_j}\left(\frac{\partial U}{\partial q_i} + \sum_\rho \frac{\partial U}{\partial q_\rho} b_{\rho i} + \sum_{\beta,\gamma} B_{i\beta\gamma}\omega_\beta\omega_\gamma\right)$$
$$v_{ij} = \sum_\beta B_{ij\beta}\omega_\beta - f_{ij}, \ u_{i\beta} = \sum_\gamma (B_{i\beta\gamma} + B_{i\gamma\beta})\omega_\gamma, \ v_{\alpha j} = \sum_\beta B_{\alpha j\beta}\omega_\beta$$
$$u_{i\sigma} = \frac{\partial}{\partial q_\sigma}\left(\frac{\partial U}{\partial q_i} + \sum_\rho \frac{\partial U}{\partial q_\rho} b_{\rho i} + \sum_{\beta,\gamma} B_{i\beta\gamma}\omega_\beta\omega_\gamma\right)$$

where the zero superscript indicates that the respective quantity is calculated for $x = z = 0$ (in input variables for $q_i = q_{i0}$ and $q_\rho = q_{\rho 0}$).

The characteristic equation of system (2.2) has $m - k$ zero roots and the remaining $2k$ roots satisfy the equation

$$\det\begin{Vmatrix} \| a_{ij}^{\circ}\lambda^2 - v_{ij}^{\circ}\lambda - u_{ij}^{\circ} \| & \| \overset{\circ}{a}_{i\beta}\lambda - \overset{\circ}{u}_{i\beta} \| & \| -\overset{\circ}{u}_{i\sigma} \| \\ \| \overset{\circ}{a}_{\alpha j}\lambda - \overset{\circ}{v}_{\alpha j} \| & \| \overset{\circ}{a}_{\alpha\beta} \| & \| 0 \| \\ \| -\overset{\circ}{b}_{\rho j} \| & \| 0 \| & \| \delta_{\rho\sigma} \| \end{Vmatrix} = 0 \tag{2.3}$$

3. If at least one root of Eq. (2.3) lies in the right-hand half-plane, solution (1.6) is unstable. Let us show that when all roots of Eq. (2.3) lie in the left-hand half-plane, we have a particular case of the critical case of several zero roots.

Equations (2.2) admit $m-k$ linear integrals

$$\sum_j \overset{\circ}{a}_{\alpha j} x_j^{\cdot} + \sum_\beta \overset{\circ}{a}_{\alpha\beta} y_\beta - \sum_j \overset{\circ}{v}_{\alpha j} x_j = \eta_\alpha \quad (\eta_\alpha = \text{const}, \alpha = k+1, \ldots, l) \tag{3.1}$$

$$z_\rho - \sum \overset{\circ}{b}_{\rho j} x_j = \zeta_\rho \ (\zeta_\rho = \text{const}, \ \rho = l+1, \ldots, m) \tag{3.2}$$

We substitute variables η and ζ for y and z using formulas (3.1) and (3.2), respectively. We first solve system (2.1) for higher derivatives, then write the equations of perturbed motion in variables x, $\xi = x^{\cdot}$, η and ζ, We obtain

$$x_i^{\cdot} = \xi_i$$

$$\xi_i^{\cdot} = \sum_j A_{ij}(x, \zeta)\Phi_j(x, \eta, \zeta) + \sum_j \Psi_{ij}(x, \xi, \eta, \zeta)\xi_j$$

$$\eta_\alpha^{\cdot} = \sum_{j,s} \overset{\circ}{a}_{\alpha s} A_{sj}(x, \zeta)\Phi_j(x, \eta, \zeta) + \sum_j \Psi_{\alpha j}(x, \xi, \eta, \zeta)\xi_j$$

$$\zeta_\rho^{\cdot} = \sum_j \beta_{\rho j}(x, \zeta)\xi_j$$

where A_{rs} are elements of matrix which is the inverse of matrix $\| a_{rs} \| (r, s = 1, \ldots, l)$, and the expansion of functions Φ_j, $\Psi_{\alpha j}$, and $\beta_{\rho j}$ in powers of its variables begins with terms of an order not lower than the first, while Ψ_{ij}^0 are generally nonzero. The explicit form of functions Φ, Ψ, and β is not presented owing to their unwieldiness.

Since the expansion of functions Φ_i may contain terms that are linear with respect to η and ζ, it is necessary to transform the variables so as to reduce the system to the standard form for the investigation of the critical case of several zero roots. For this we shall consider the system of equations

$$\xi_i = 0, \quad \sum_j A_{ij}(x, \zeta)\Phi_j(x, \eta, \zeta) + \sum_j \Psi_{ij}(x, \xi, \eta, \zeta)\xi_j = 0 \quad (i = 1, \ldots, k)$$

by solving which for x and ξ we obtain

$$\xi_i = 0, \ x_i = X_i(\eta, \ \zeta)$$

where X_i satisfy the system of equations

$$\Phi_j(X, \eta, \zeta) = 0 \quad (j = 1, \ldots, k) \ (\det \| A_{ij} \| \neq 0)$$

whose solution is known to exist, since it is assumed that all roots of Eq. (2.3) lie in the left half-plane.

We carry out the change of variables

$$x_i = X_i(\eta, \ \zeta) + \chi_i$$

and write the equations of perturbed motions in variables χ, ξ, η, and ζ

$$\chi_i^{\cdot} = \xi_i + \Xi_i(\chi, \ \xi, \ \eta, \ \zeta) \tag{3.3}$$

$$\xi_i^{\cdot} = \sum_j \Big[\sum_h A_{ij}^{\circ} \Big(\frac{\partial \Phi_j}{\partial x_h}\Big)_0 \chi_h + \Psi_{ij}^{\circ}\xi_j \Big] + \Xi_{k+i}(\chi, \xi, \eta, \zeta)$$

$$\eta_\alpha^{\cdot} = \sum_{j,s} \overset{\circ}{a}_{\alpha s} A_{sj}(X(\eta, \zeta) + \chi, \zeta)\Phi_j(X(\eta, \zeta) + \chi, \eta, \zeta) + \sum_j \Psi_{\alpha j}(X(\eta, \zeta) + \chi, \xi, \eta, \zeta)\xi_j$$

$$\zeta_\rho^{\cdot} = \sum_j \beta_{\rho j}(X(\eta, \zeta) + \chi, \zeta)\xi_j$$

$$\Xi_i = \Big[\sum_\alpha \frac{\partial X_i}{\partial \eta_\alpha}\eta_\alpha^{\cdot} + \sum_\rho \frac{\partial X_i}{\partial \zeta_\rho}\zeta_\rho^{\cdot} \Big]_{(3.3)} = \Xi_i(\chi, \xi, \eta, \zeta)$$

$$\Xi_{k+i} = \sum_j \Big\{ A_{ij}(X(\eta, \zeta) + \chi, \zeta)\Phi_j(X(\eta, \zeta) + \chi, \eta, \zeta) - \sum_h A_{ij}^{\circ}\Big(\frac{\partial \Phi_j}{\partial x_h}\Big)_0 \chi_h + [\Psi_{ij}(X(\eta, \zeta) + \chi, \xi, \eta, \zeta) - \Psi_{ij}^{\circ}]\,\xi_j \Big\}$$

When $\chi = \xi = 0$ the right-hand sides of equations for $\eta^{\cdot}$ and $\zeta^{\cdot}$ of system (3.3) are identically zero, hence also $\Xi_i(0, 0, \eta, \zeta) = \Xi_{k+i}(0, 0, \eta, \zeta) \equiv 0$ /3,4/, i.e. system (3.3) is of the form that corresponds to the particular case of the critical case of several zero roots. Thus, when all roots of Eq. (2.3) lie in the left half-plane, the Liapunov--Malkin theorem holds and solution (1.6) is stable (but not asymptotically). Then any perturbed motion fairly close to the unperturbed approaches one of the possible steady motions of the form (1.6) that belong to manifold (1.7) (but not to the unperturbed one) as $t \to +\infty$.

4. Using elementary transformations it is possible to reduce Eq. (2.3) the form (where the prime denotes transposition)

$$\det \| A\lambda^2 - (G + D)\lambda - (C + E) \| = 0 \tag{4.1}$$

$$A = \| a_{ij}^\circ - \sum_{\alpha,\beta} \overset{\circ}{h}_{\alpha\beta} \overset{\circ}{a}_{\alpha i} \overset{\circ}{a}_{\beta j} \|, \quad \| \overset{\circ}{h}_{\alpha\beta} \| = \| \overset{\circ}{a}_{\alpha\beta} \|^{-1} \ (\alpha, \beta = k+1, \ldots, l)$$

$$G + D = \| v_{ij}^\circ - \sum_{\alpha,\beta} \overset{\circ}{h}_{\alpha\beta} (\overset{\circ}{a}_{i\beta} \overset{\circ}{v}_{\alpha j} + \overset{\circ}{a}_{\alpha j} \overset{\circ}{u}_{i\beta}) \|, \quad G' = -G, \ D' = D$$

$$C + E = \| u_{ij}^\circ + \sum_{\alpha,\beta} \overset{\circ}{h}_{\alpha\beta} \overset{\circ}{u}_{i\beta} \overset{\circ}{v}_{\alpha j} + \sum_{\rho} \overset{\circ}{u}_{i\rho} \overset{\circ}{b}_{\rho j} \|, \quad C' = C, \ E' = -E$$

Equation (4.1) may be considered to be the characteristic equation of the system of equations of motion

$$Aw^{\cdot\cdot} = Gw^{\cdot} + Dw^{\cdot} + Cw + Ew \tag{4.2}$$

where w is a k-dimensional column vector, of the holonomic system with k degrees of freedom, subjected to the action of potential, position nonpotential, gyroscopic, and dissipative-accelerating forces. It is possible to reduce the problem of steady motion stability of the input system, as in /1/, to that of equilibrium stability of system (4.2) which we shall call "reduced". As in /1/ it is possible to obtain a number of theorems on stability, or instability of steady motions of nonholonomic systems of the general form. All results of /1/ stated in the form of conditions on matrices appearing in Eq. (4.1) remain valid (the form of these matrices, except matrix A, changes but, generally, coincides with the form in /1/ only when $m = l$).

Let us now assume that external dissipative forces are absent ($F \equiv 0$). In this case matrices A, G, C, and E remain unchanged, while matrix D does not vanish and assumes the form

$$D^* = \| d_{ij}^* \|; \ d_{ij}^* = \frac{1}{2} \sum_{\beta} \omega_\beta \Big\{ \sum_{\mu} (\overset{\circ}{\theta}_{\mu j} \overset{\circ}{\nu}_{\mu i\beta} + \overset{\circ}{\theta}_{\mu i} \overset{\circ}{\nu}_{\mu j\beta}) - \sum_{\mu,\alpha,\gamma} \overset{\circ}{h}_{\alpha\gamma} [\overset{\circ}{\nu}_{\mu\alpha\beta} (\overset{\circ}{a}_{i\gamma} \overset{\circ}{\theta}_{\mu j} + \overset{\circ}{a}_{j\gamma} \overset{\circ}{\theta}_{\mu i}) + \overset{\circ}{\theta}_{\mu\gamma} (\overset{\circ}{a}_{j\alpha} \overset{\circ}{\nu}_{\mu i\beta} + \overset{\circ}{a}_{i\alpha} \overset{\circ}{\nu}_{\mu j\beta})] \Big\}$$

This shows that under specific conditions Eq. (4.1) can have all of its roots in the left half-plane also for conservative nonholonolic systems.

The above evidently occurs, for instance, under conditions /5/

$$E \equiv 0; \ w'(-C)w > 0, \ \forall w \neq 0; \ w'(-D^*)w > 0, \ \forall w \neq 0 \tag{4.3}$$

or under conditions /6,7/

$$w'(-C)w > 0, \ \forall w \neq 0; \ -D^* = \delta D_*, \ w' D_* w > 0, \ \forall w \neq 0 \tag{4.4}$$

($\delta > 0$ is fairly large) which means that in the reduced system the potential energy has a minimum in equilibrium, the dissipative-accelerating forces are pure dissipative with total dissipation and, either potential position forces are absent (4.3), or the dissipative forces of the reduced system are fairly intensive (4.4).

5. We thus have the following position the characteristic equation of the system of equations for perturbed motion in the neighborhood of steady motion of a conservative nonholonomic system has, under certain conditions (e.g., (4.3) or (4.4)), besides zero roots, all its remaining roots with negative real parts. This phenomenon, impossible for holonomic systems, has a fairly simple explanation: the equations of perturbed motion of nonholonomic systems contain also in the conservative case besides the input forces, other forces including dissipative-accelerating ones. To verify this it is sufficient to write down the nonholonomic terms of the input system in the neighborhood of solution (1.6). Note that the nonholonomy terms in the equations of motion are linear combinations of reactions of nonholonomic relations, quadratic forms with respect to velocities, which in the steady motion neighborhood are sums of potential, nonpotential positional, gyroscopic, and dissipative accelerating forces. The last of which may under certain conditions be purely dissipative, yielding the unexpected effect of the presence in a conservative nonholonomic systems of roots of the characteristic equation with negative real parts and absence roots with positive real parts.

This means, among other things, that, although the input system and the equations of perturbed motion admit in the conservative case an energy integral of the form

$$\Theta - U = \text{const} \tag{5.1}$$

the linearized system does not admit such integral. Function Θ in (5.1) is a quadratic form of velocities $q_s^{\cdot}$ $(s = 1, \ldots, l)$ or the sum of quadratic and linear forms of velocity perturbations $x_i^{\cdot}$ $(i = 1, \ldots, k)$ and y_α $(\alpha = k+1, \ldots, l)$, as well as the function of variables x_i and z_ρ independent of $x_i^{\cdot}$ and y_α, which in variables χ, ξ, η, and ζ contains terms of the form

$$\sum_\alpha \omega_\alpha \eta_\alpha \tag{5.2}$$

The total derivative of function (5.2) is by virtue of the linearized system zero, while by virtue of the total system it is nonzero, and contains terms which yield zero only in conjunction with the respective terms of the total derivative of the remaining part of function (5.1) which is nonzero by virtue of the linearized system. Perturbations of ignorable coordinate momenta can be juxtaposed to variables η_α in holonomic systems. The ignorable coordinates maintain their initial values and by virtue of the complete system of equations of perturbed motion, since in holonomic systems the first integrals correspond to ignorable coordinates.

Note that the coefficients of matrix D^* linearly depend on the velocities of ignorable coordinates so that under specific conditions with a certain selection of signs of ω_α $(\alpha = k + 1, \ldots, l)$ the dissipative accelerating forces may be purely dissipative, while with the opposite selection of signs of ω_α be purely accelerating. This means that the steady motions of nonholonomic systems can be stable (asymptotically with respect to variables χ and ξ) when the motion is "in one direction" and unstable when it is in the "opposite direction" (under otherwise equal conditions).

6. Example /8/. Consider a heavy solid body bounded by a convex surface laying on a horizontal absolutely rough plane. The position of the body is specified by the x- and y-coordinates of its center of mass in the $Oxyz$ system of coordinates (the plane Oxy coincides with the horizontal plane, and the Oz-axis is directed vertically up) and by Euler's angles θ, φ, and ψ formed by the principal central axes $G\xi$, $G\eta$, and $G\zeta$ of the body ellipsoid of inertia, and the axes of the stationary coordinate system. Then the Lagrange function and the system relations which express the absence of slip at the point of contact of the body with the plane assume the form

$$L = \frac{1}{2}[A\cos^2\varphi + B\sin^2\varphi + m(\gamma_1\cos\theta - \zeta\sin\theta)^2]\,\theta^{\cdot 2} + \frac{1}{2}(C + m\gamma_2^2\sin^2\theta)\,\varphi^{\cdot 2} + \frac{1}{2}[(A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + C\cos^2\theta]\,\psi^{\cdot 2} +$$
$$m(\gamma_1\cos\theta - \zeta\sin\theta)\gamma_2\sin\theta\,\theta^{\cdot}\varphi^{\cdot} + (A - B)\sin\theta\sin\varphi\cos\varphi\,\theta^{\cdot}\psi^{\cdot} + C\cos\theta\,\varphi^{\cdot}\psi^{\cdot} + \frac{1}{2}m(x^{\cdot 2} + y^{\cdot 2}) + mg(\gamma_1\sin\theta + \zeta\cos\theta)$$

$$x^{\cdot} = \alpha_1\theta^{\cdot} + \alpha_2\varphi^{\cdot} + \alpha_3\psi^{\cdot}, \quad y^{\cdot} = \beta_1\theta^{\cdot} + \beta_2\varphi^{\cdot} + \beta_3\psi^{\cdot}$$
$$\alpha_1 = -(\gamma_1\sin\theta + \zeta\cos\theta)\sin\psi, \quad \alpha_2 = \gamma_2\cos\theta\sin\psi + \gamma_1\cos\psi$$
$$\alpha_3 = \gamma_2\sin\psi + (\gamma_1\cos\theta - \zeta\sin\theta)\cos\psi, \quad \beta_i = -\partial\alpha_i/\partial\psi \quad (i = 1, 2, 3)$$
$$\gamma_1 = \xi\sin\varphi + \eta\cos\varphi, \quad \gamma_2 = \xi\cos\varphi - \eta\sin\varphi$$

where m is the mass of the body; A, B, and C are its principal central moments of inertia; ξ, η, ζ are the coordinates of the point of contact of the body and the horizontal plane in the system $G\xi\eta\zeta$. It can be shown that ξ, η, and ζ are functions of variables θ and φ determined by the form of the equation which specifies the boundary surface of the body.

The above system is obviously a Chaplygin system; a direct test of conditions (1.4) will prove that ψ is an ignorable coordinate. Hence the input system admits the solution

$$\theta = \text{const}, \quad \varphi = \text{const}, \quad \psi^{\cdot} = \text{const}$$

and, in particular, the solution

$$\theta = \pi/2, \quad \varphi = 0, \quad \psi^{\cdot} = \omega = \text{const} \tag{6.1}$$

which corresponds to rotation of the body at constant angular velocity about the vertical principal axis $G\eta$ of the body ellipsoid of inertia.

Omitting the presentation of the input equations of motion because of their unwieldiness, we write the equations of perturbed motion

$$(A + ma^2)u^{\cdot\cdot} = ma(r_2 - r_1)\sin\alpha\cos\alpha\,\omega u^{\cdot} - [A + C - B + 2ma^2 - ma(r_1\sin^2\alpha + r_2\cos^2\alpha)]\omega v^{\cdot} + [(C - B)\omega^2 + m(a - r_1\cos^2\alpha - r_2\sin^2\alpha)(g + a\omega^2)]\,u - m(r_2 - r_1)\sin\alpha\cos\alpha\,(g + a\omega^2)\,v + U$$

$$(C + ma^2)v^{\cdot\cdot} = [A + C - B + 2ma^2 - ma(r_1\cos^2\alpha + r_2\sin^2\alpha)]\omega u^{\cdot} - ma(r_2 - r_1)\sin\alpha\cos\alpha\,\omega v^{\cdot} - m(r_2 - r_1)\sin\alpha\cos\alpha\,(g + a\omega^2)u + [(A - B)\omega^2 + m(a - r_1\sin^2\alpha - r_2\cos^2\alpha)(g + a\omega^2)]v + V$$

$$Bw^{\cdot} = W$$

where u, v, and w are perturbations of variables θ, φ, and $\psi^{\cdot}$; U, V, and W are functions of variables u, $u^{\cdot}$, v, $v^{\cdot}$, and w whose expansion begins with terms of order not lower than the second, and $U(0, 0, 0, 0, w) = V(0, 0, 0, 0, w) = W(0, 0, 0, 0, w) \equiv 0$; a is the distance between the point of contact of the body with the plane, and its center of mass; r_1 and r_2 are the principal curvature radii of the body surface at the point of its contact with the plane; and α is the angle between the principal axis of the central ellipsoid of inertia at instant $C(G\zeta)$ and the direction of the principal radius of curvature r_1.

Composing the characteristic equation for the linearized system, rejecting the zero root that corresponds to the critical variable w, and applying the described above results, we obtain that the steady motion (6.1) is stable and, when all roots of the equation

$$P\lambda^4 + Q\omega\lambda^3 + R\lambda^2 + Q\omega^3\lambda + S = 0 \qquad (6.2)$$

lie in the left half-plane, it is asymptotically stable with respect to $\theta, \theta^{\cdot}, \varphi$, and $\varphi^{\cdot}$; it is unstable, if at least one of the roots of Eq. (6.2) lies in the right half-plane. In this equation

$$P = (A + ma^2)(C + ma^2), \quad Q = (A - C)\, ma\, (r_2 - r_1) \sin\alpha\cos\alpha$$

$$R = [(A + C - B + 2ma^2)^2 - (A + C - B + 2ma^2)\, ma\, (r_1 + r_2) + m^2a^2r_1r_2]\omega^2 - (A + ma^2)[(A - B)\omega^2 + m\,(a - r_1\sin^2\alpha - r_2\cos^2\alpha)\cdot (g + a\omega^2)] - (C + ma^2)[(C - B)\omega^2 + m\,(a - r_2\sin^2\alpha - r_1\cos^2\alpha)(g + a\omega^2)]$$

$$S = (A - B)(C - B)\omega^4 + m\,(g + a\omega^2)\omega^2\,[A\,(a - r_1\cos^2\alpha - r_2\sin^2\alpha) + C\,(a - r_1\sin^2\alpha - r_2\cos^2\alpha) - B\,(2a - r_1 - r_2)] + m^2\,(g + a\omega^2)^2\cdot (a - 2_1)(a - r_2)$$

Using the Hurwitz criterion we find that all roots of Eq. (6.2) lie in the left half-plane, if the conditions

$$(R - P\omega^2)\omega^2 - S > 0, \quad S > 0 \qquad (6.3)$$

$$(A - C)(r_2 - r_1)\,\omega\sin\alpha\cos\alpha > 0 \qquad (6.4)$$

are satisfied. If even only one of these inequalities is violated, Eq. (6.2) has at least one root in the right half-plane.

Inequalities (6.3) impose constraints only on mass distribution in the body, and its surface geometry, and on the angular velocity, while inequality (6.4) imposes the constraint on the sign of angular velocity (direction of rotation). The last inequality has a simple geometrical interpretation. The rotation is stable when a principal axis of inertia of the body (the large or the small) precedes the respective (large or small) axis of principal curvatures of the body surface at the point of its contact with the horizontal plane, and is unstable in the opposite case /8/. Another interpretation was proposed by Rumiantsev. The quantity $(A - C)\sin\alpha\cos\alpha$ represents the product of inertia of the body with respect to axes linked to the directions of the principal radii of curvature of the body surface. Condition (6.4) then indicates that, for instance, when $r_2 > r_1$, in the case of stability the sign of the product of inertia must be the same as that of the angular velocity of rotation of the body.

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